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Assessment of Damage Aggregation for Weapons Salvos

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Estimation of the expected proportion of an area target destroyed by a salvo of weapons is an important problem in weapons systems effectiveness studies. Analysts sometimes use an approximate formula called the "empirical rule" for this purpose. Esary [1], [2] demonstrated that this rule consistently over estimates the parameter of interest. In this report, we propose a modification to the empirical rule that appears to provide significantly improved estimates of the cumulative damage.

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ASSESSMENT OF DAMAGE AGGREGATION FOR WEAPONS SALVOS

Toke Jayachandran and I.Bert Russak

1. INTRODUCTION. Estimation of the expected proportion of an area target damaged by a salvo of weapons is an important problem in weapons systems effectiveness studies. Analysts sometimes use an approximate extant formula, referred to herein as the "empirical rule", for this purpose. The genesis and the rationale underlying this formula are not readily available. Esary [1] , [2] introduced a "proportional effects model" for which the empirical formula is a first-order approximation to the expected cumulative damage to the target. Assessment of this damage exactly would require a knowledge of the probability distribution of the number of weapons from the salvo that will impact on the target. Esary investigated the accuracy of the empirical rule approximation for three choices of the weapons impact distribution. His conclusion was that the empirical rule consistently overestimates the cumulative damage for all possible impact distributions and that the approximation can be quite poor for many realistic situations. The advantage in using the empirical rule is that one can compute, approximately, the cumulative damage using only the mean or expected number of weapons that will hit the target, without knowing the actual distribution of hits. In this report, we propose modifications to the empirical rule that use in addition to the mean, the variance of the hit distribution. These modifications result in significantly improved measures of the expected cumulative damage to the target.

2. PROBLEM DESCRIPTION. A salvo of n weapons is launched at an area target. The number of weapons that will actually hit the target is a random variable N whose possible values are 0,1,2,. . .,n. Each hit will result in damaging a certain proportion of the target. The cumulative damage to the target is then a random variable D which depends on N the random number of weapons that impact the target. The quantity of interest is E(D) the expected (or mean) cumulative damage. To estimate E(D) exactly, the probability distribution of N the number of weapons that impact the target, the target configuration and the damage aggregation scenario for multiple hits are needed. However, in practice analysts are known to use the empirical rule

$$E(D) \quad = \quad 1 - (1\text{-}d)^{E(N)} \qquad (1)$$

where d is the damage proportion for a single hit and E(N) is the expected number of weapons out a salvo of n weapons that hit the target. As mentioned earlier, the genesis of this formula is not well documented. Esary [1] considered the following “proportional effects” model that can be used to provide a rationale for the empirical rule. The premise of this model is that a proportion d of a pristine target is damaged by the first hit and each additional hit damages the same proportion d of that part of the target not previously damaged. Then, it can be shown that if N weapons hit the target, the cumulative damage is

$$D(N) \quad = \quad 1 - (1\text{-}d)^{N} \qquad (2)$$

and E[D(N)] , or simply E(D), is the expected cumulative damage to the target. Replacing N with E(N) in D(N) will result in the empirical rule E(D) which will be different from E(D), in general. To evaluate E(D) exactly, the probability distribution of N, the number of weapon hits is needed. Esary proved that the empirical rule consistently over estimates the expected cumulative damage i.e., E(D) > E(D) for every possible hit distribution. If it is assumed that the distribution is Binomial i.e., if each of the n weapons hits the target independently with the same probability p, the expected cumulative damage

$$E_B(D) \quad = \quad 1 - (1\text{-}pd)^{n} \qquad (3)$$

and the estimate using the empirical rule for this case is

$$E_B(D) \quad = \quad 1\text{-} (1\text{-}D)^{np} \qquad (4)$$

For a Poisson (sometimes used as an approximation to the Binomial) hit distribution with parameter λ=np,

$$E_P(D) \quad = \quad 1 - e^{-\lambda d} \qquad (5)$$

and $E_P(D)$ will be the same as for the binomial distribution. The corresponding formulas for a Uniform hit distribution are

$$E_U(D) \quad = \quad 1 - (1\text{-}d)^{n+1}/(n+1)d \qquad (6)$$

$$E_U(D) \quad = \quad 1 - (1-d)^{n/2}. \tag{7}$$

For the Binomial and the Poisson distributions, the exact values of E(D) were compared with the corresponding approximate values for the empirical rule, for various choices of n, p and d, to demonstrate the magnitudes of the errors in the approximation. The resulting graphs are presented in figures 1-16. It can be seen that the two measures are reasonably close for small values of d ($0<d<.5$) and they diverge markedly as d approaches 1 for all choices of n and p. However, the disparity decreases as n the number of weapons in the salvo increases.

3. <u>A SECOND-ORDER TAYLOR APPROXIMATION</u>. The Taylor approximating polynomial of degree two about the point $X=\mu_X$ for a function f(X) of a random variable X with $\mu_X = E(X)$ is

$$f(X) = D(\mu_X) + f'(\mu_X)(X-\mu_X) + f''(\mu)(X-\mu_X)^2/2 \tag{8}$$

and taking expectations on both sides

$$E[f(X)] = f(\mu_X) + f''(\mu_X)\,\sigma^2{}_X/2 \tag{9}$$

where $\sigma^2{}_X = V(X)$ the variance of the random variable X. Applying this result to D(N) in equation (2) above, we get the second-order Taylor approximation to E[D(N)],

$$T[D(N)] = D(\mu_N) + D''(\mu_N)\,\sigma^2{}_N/2 \tag{10}$$

$$=1-\left\{1+[\ln(1-d)]^2\sigma^2{}_N/2\right\}(1-d)^{\mu_N}$$

where ln is the natural logarithm function and μ_N, $\sigma^2{}_N$ are the mean and the variance of the random variable N. We propose T[D(N)], the Taylor approximation as a better alternative to the empirical rule in equation (1). Whereas only μ_N is needed to use the empirical rule, both μ_N and $\sigma^2{}_N$ are required for the use of this second-order approximation.

Graphs of the Taylor approximation are included in figures 1-16; they show significant improvements in the approximation relative to the empirical rule, for all choices of the parameters n,p,d. However, for

larger values of d ($d>.8$) the Taylor approximation errors are still a little on the high side (albeit smaller than that for the empirical rule), especially for small values of n the number of weapons in the salvo. To remedy this problem, we propose an ad hoc correction, determined through trial and error, to the Taylor approximation, to further reduce the error at the high values of d. Specifically, we suggest the modified formula

$$M[D(N)] = T[D(N)] - D^4(1-P^2)/2(N+1) \ . \qquad (11)$$

Approximation error plots for the modified formula are also shown in figures 1-16. The modification appears to work well, in most cases, as can be seen from the closeness of the graphs to those for the exact formula. The exceptions are for the case of the Uniform hit distribution which is a flat distribution i.e., its graph is rectangular in shape. The Uniform distribution as a model for the number of hits in a salvo of n weapons is not very realistic since it would imply that the probability of 0 hits or 1 hit or 2 hits . . . or n hits is exactly the same. The distribution was included in this study to indicate the behavior of the approximations in the case of a relatively flat hit distribution.

Error percentages (100x(approximate - exact)/exact) are presented in Tables 1-3 for each of the three approximation formulas viz., Empirical, Taylor and Modified for n=4,6,8; p=.4,.6,.8; and the three hit distributions, Binomial, Poisson and the Uniform. These tables show more clearly the improvements attained by using the approximations proposed in this paper.

4. <u>CONCLUSIONS</u>. As shown by Esary, the empirical rule grossly overestimates the expected cumulative damage. We proposed two new approximate rules viz., the Taylor approximation and the Modified approximation that consistently provide better results than those for the empirical rule. The modified approximation provides the best measures unless the hit distribution is somewhat flat, i.e, the distribution is close to the uniform distribution; in which case the Taylor approximation appears to be the best choice.

TABLE I

PERCENT ERRORS FOR THE BINOMIAL DISTRIBUTION

N	p \ d	EMPIRICAL .4	.6	.8	.95	.99	TAYLOR .4	.6	.8	.95	.99	MODIFIED .4	.6	.8	.95	.99
4	.4	11	15	18	16	15	2	2	6	12	15	0	0	1	4	5
	.6	6	7	6	4	3	1	2	3	3	3	0	1	0	-2	-4
	.8	2	7	1	0	0	0	1	1	0	0	0	0	-1	-3	-3
6	.4	9	10	9	6	5	0	2	4	5	5	0	1	2	0	0
	.6	4	3	2	1	1	1	1	1	1	1	0	0	-1	-3	-4
	.8	1	1	0	0	0	0	0	0	0	0	0	0	0	-2	-3
8	.4	7	7	4	2	2	1	2	3	2	2	0	1	1	-2	-3
	.6	3	2	1	0	0	0	1	0	0	0	0	0	-1	-3	-3
	.8	1	0	0	0	0	0	0	0	0	0	0	0	-1	-2	-2

TABLE II

PERCENT ERRORS FOR THE POISSON DISTRIBUTION

N		EMPIRICAL					TAYLOR					MODIFIED				
	d / p	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99	.4	.6	.8	.95	.99
4	.4	18	25	28	27	26	6	10	15	22	25	6	8	10	14	15
	.6	15	17	15	11	10	9	11	12	11	10	8	10	9	5	3
	.8	12	11	8	5	4	9	10	7	5	4	9	9	6	2	1
6	.4	15	17	15	11	10	6	8	10	10	10	5	7	7	5	3
	.6	10	9	6	3	3	6	6	5	3	3	6	6	3	-1	-2
	.8	7	5	2	1	0	6	4	2	1	1	6	4	-1	-1	-2
8	.4	12	11	8	5	4	5	6	6	5	4	5	5	4	1	0
	.6	7	5	2	1	1	5	4	2	1	1	5	3	1	-2	-3
	.8	4	2	1	0	0	4	2	1	0	0	4	2	0	-1	-2

TABLE III

PERCENT ERRORS FOR THE UNIFORM DISTRIBUTION

N	d / p	EMPIRICAL .4	.6	.8	.95	.99	TAYLOR .4	.6	.8	.95	.99	MODIFIED .4	.6	.8	.95	.99
4	.4	19	25	28	26	25	1	6	14	24	25	1	4	10	15	15
	.6	19	25	28	26	25	1	6	14	24	25	1	4	11	17	17
	.8	19	25	28	26	25	1	6	14	24	25	1	5	12	20	21
6	.4	20	23	21	18	17	3	9	16	17	17	3	8	13	12	10
	.6	20	23	21	18	17	3	9	16	17	17	3	8	13	13	12
	.8	20	23	21	18	17	3	9	16	17	17	3	8	14	15	14
8	.4	20	20	16	13	13	5	11	14	13	13	4	10	12	9	8
	.6	20	20	16	13	13	5	11	14	13	13	4	10	13	10	9
	.8	20	20	16	13	13	5	11	14	13	13	4	10	13	11	11

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[2] J.D.Esary. *Studies On Damage Aggregation For Weapons Salvos II,* Naval Postgraduate School Technical Report NPSOR-92-007, December 1991.

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